Correspondence

A Note on Matrix Powers

***Abstract*—For an $m \times m$ matrix T a systematic procedure is given for expressing $T^N (N \geq m)$ as a linear combination of powers of T less than m.**

Recently Rao and Ahmed [1] wrote concerning the powers of a 2×2 matrix. Although it was not necessary, they restricted themselves to matrices with nonunity determinants. In this correspondence a systematic procedure will be described for calculating the powers of any square matrix. The general procedure will be applied to matrices with distinct eigenvalues (including zero). Modifications and alternatives for the case of multiple eigenvalues are presented also.

Let T be an $m\times m$ matrix with distinct eigenvalues $\mu_1, \mu_2, \ldots \mu_m$. The eigenvalues satisfy the characteristic equation

$$f(\lambda) = \prod_{i=1}^{m} (\lambda - \mu_i) = \sum_{i=0}^{m} a_i \lambda^{m-i} = 0. \quad (1)$$

The Cayley–Hamilton theorem tells us that

$$f(T) = 0. \quad (2)$$

Since $f(\lambda)=0$, any λ^N with $N \geqq m$ can be written as a linear combination of the powers λ^i where $i<m$. Thus λ^N can be given as

$$\lambda^N = \sum_{i=0}^{m-1} b_i \lambda^{m-1-i} \quad (N \geqq m). \quad (3)$$

From (2) T^N can also be written in the form of (3) as

$$T^N = \sum_{i=0}^{m-1} b_i T^{m-1-i} \quad (N \geqq m). \quad (4)$$

The problem then is to determine b_i in (3). This can be accomplished by successively inserting the eigenvalues μ_i in (3). The following matrix equation results.

$$\begin{bmatrix} \mu_1^N \\ \mu_2^N \\ \vdots \\ \mu_m^N \end{bmatrix} = \begin{bmatrix} \mu_1^{m-1} & \mu_1^{m-2} & \cdots & \mu_1 & 1 \\ \mu_2^{m-1} & \mu_2^{m-2} & \cdots & \mu_2 & 1 \\ \vdots & \vdots & & & \vdots \\ \mu_m^{m-1} & \mu_m^{m-2} & \cdots & \mu_m & 1 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{m-1} \end{bmatrix} \quad (N \geq m). \quad (5)$$

This relation may be inverted since the $m\times m$ matrix on the right is nonsingular; its determinant is a Vandermonde [2] and is nonzero for distinct μ_i. In (4) b_i may thus be determined uniquely. For a 2×2 matrix $T(m=2)$ the inversion of (5) yields

$$b_0 = \frac{\mu_1^N - \mu_2^N}{\mu_1 - \mu_2} \quad (6a)$$

$$b_1 = \frac{\mu_2^N \mu_1 - \mu_1^N \mu_2}{\mu_1 - \mu_2} \quad (6b)$$

which of course agrees with previous results [1], [3].

If T has some multiple eigenvalues, T^N may be found in closed form by applying L'Hospital's rule (repeatedly if necessary) to (3) after (5) is inverted (assuming distinct eigenvalues); or, (1) may be used to solve for λ^N in closed form by mathematical induction.

Often a savings in computation can be achieved by using the minimum polynomial $m(\lambda)$ [2], instead of the characteristic polynomial $f(\lambda)$, in (1) and thereafter. $m(\lambda)$ will contain all the linear factors of $f(\lambda)$ but may be considerably reduced in degree if multiple eigenvalues are present.

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An Optical Data-Processing Experiment for Electrical Engineering Students

***Abstract*—An experiment which optically demonstrates the principles of electronic signal filtering is reviewed. A lens forms the Fourier transform of a transparency at its back focal plane. Then, by proper placement of stops and apertures at the focal plane, the image is altered in a manner analogous to electronic filtering.**

The increased number of industrial applications of lasers is creating a need for what might be termed electrooptical engineers. Therefore, to ensure that electrical engineering graduates have some knowledge of lasers and laser systems, a series of project-type experiments have been developed for undergraduate students [1]. These experiments are designed to be incorporated into existing undergraduate-laboratory programs. This enables the student to learn about lasers without taking a special course in electrooptics. Each of the projects is based on some current, industrial laser application and can be performed using equipment costing less than $500 (including the laser).

As an example of one of the projects, the student might be asked to investigate optical spatial filtering. When a monochromatic plane wave from a laser is incident on a convex lens, the field pattern at the back focal plane of the lens is proportional to the Fourier transform of the field pattern of the incident light [2]. This Fourier transform is a two-dimensional transform in space rather than the usual one-dimensional transform in time. The transform relationship between fields on either side of a lens thus enables one to modify the signal by operating on its Fourier spectrum that appears at the back focal plane.

A typical equipment arrangement for performing optical data-processing experiments is shown in Fig. 1. The combination of lenses L_1, L_2, and L_3 expand and collimate the laser beam. The signal to be operated on is inserted in the input plane x_s by means of Polaroid film transparencies. Standard Polaroid 146 L film makes transparencies that are quite satisfactory. Lens L_4 forms the two-dimensional Fourier transform of the signal at x_f, which is called the frequency plane. Then L_5 performs the inverse Fourier transform on the signal at x_f, and the signal appears again (but inverted) at the screen S.

One impressive demonstration of spatial filtering is the removal of dots from a newspaper halftone photograph (a newspaper picture formed by an array of closely spaced dark dots of varying size). Suppose a transparency of a newspaper picture is inserted at the plane x_s. The image formed at x_f is a bright dot on the optical axis with a series of less intense dots surrounding the center dot. The spot at the center corresponds to the low frequency portions of the photograph while the spots around the center correspond to the higher order frequencies caused by the dark dots on the photograph. The spots further from the center correspond to the higher frequency components.

The image formed on the screen (shown in Fig. 2) is the same as the image on the transparency. However, if a pinhole is inserted at x_s so that only the low spatial frequency terms pass through the opening, the image formed on the screen is the same as before—except that the dots that formed the original image are not present. Fig. 3 shows the image of the face without the dots. The image is distorted because some of the information about the face was blocked along with the information about the dots. However, by optimizing the size of the pinhole, this distortion should be negligible. The pinhole used in this experiment was made by simply perforating a piece of mylar with a highly sharpened nail [1]. The diameter of this "home-made" pinhole was about 0.5 mm. By reversing the process and placing a stop over the low-frequency components and passing the higher frequency components, the dots appear (as shown in Fig. 4) without the image they formed previously.

This type of spatial filtering has been applied to removing the scan lines from pictures transmitted to earth by space vehicles [3]. By inserting different transparencies in the planes x_s and x_f, convolution, differentiation, correlation, antenna array simulation, and pattern recognition can be performed [4].

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