That is,

$$\begin{aligned} h_0 &= 1 \\ h_1 &= -b_1 \\ h_2 &= b_1^2 - b_2 \\ h_3 &= -b_1^3 + 2b_1b_2 - b_3 \\ &\vdots \end{aligned} \tag{7}$$

Here, $(-1)^i b_j$ is the sum of $\binom{n}{j}$ products of λ_i taken j at a time without repetition. By way of illustration, consider $n=3$. Then

$$\begin{aligned} b_1 &= -(\lambda_1+\lambda_2+\lambda_3) \\ b_2 &= \lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1 \\ b_3 &= \lambda_1\lambda_2\lambda_3. \end{aligned} \tag{8}$$

A simple check shows that the relationships among the h_i, b_j, and λ_j as indicated by (4), (5), and (8) are consistent.

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Comments on "The Mechanics of the Bilinear Transformation"

***Abstract*—Rules are given which permit the product matrix *GS*, encountered in a recent vector-matrix formulation of the effect of the transformation $z=(s+1)/(s-1)$ on a polynomial equation, to be formed directly.**

In the above paper[1] the author introduced a vector-matrix formulation of the effect of the bilinear transformation substitution

$$z = \frac{s+1}{s-1}, \tag{1}$$

on a polynomial equation

$$a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 = 0. \tag{2}$$

With the notation and equation numbering previously used, the result was

$$\boldsymbol{b} = \frac{2}{p_n}\boldsymbol{GFp}. \tag{7}$$

Despite the simple rules given for forming the $n\times(n+1)$ and $(n+1)\times n$ matrices $\boldsymbol{G}$ and $\boldsymbol{F}$, it was felt that a certain amount of unnecessary labor was involved in having to calculate the $n\times n$ product $\boldsymbol{GF}$ *ab initio* for each problem. A study has since been made of the structure of $\boldsymbol{GF}$, and the results are given below. Algebraic proofs have not yet been found for all features, but they have been verified by computation for all values of n up to 15. Any suggestions for proofs will be gratefully received!

With the first row of $\boldsymbol{F}$ chosen as

$$\left(0, 0, \cdots, 0, \frac{p_n}{2}\right),$$

the only terms in $\boldsymbol{GF}$ which contain p_n occur in the last column. All other elements are dependent only on the value of n. It has been convenient, therefore, to consider the structure of $\boldsymbol{GF}$ in two stages:

a) the $n\times n$ matrix $\boldsymbol{X}$ obtained by setting $p_n=0$;
b) the terms in p_n which must be added to the final column of $\boldsymbol{X}$ to get $\boldsymbol{GF}$.

The following features of $\boldsymbol{X}$ follow at once from the structures of $\boldsymbol{G}$ and $\boldsymbol{F}$:

1) $x_{1j} = 1$.

2) $$x_{j1} = (-1)^{j-1}\frac{(n-1)!}{(n-j)!(j-1)!} \qquad (j = 1, 2, \cdots, n).$$

Property 2 simply means that the first column of $\boldsymbol{X}$ is equal to the second column of $\boldsymbol{G}$. The following additional properties have been found by experiment:

3) $x_{jn} = |x_{j1}|$.

4) $$x_{ij} = x_{i,j+1} - x_{i-1,j+1} - x_{i-1,j} \qquad (i = 2, 3, \cdots, n;\ j = 1, 2, \cdots, n).$$

Once the first row and last column have been laid out using rules 1 and 3, rule 4 enables $\boldsymbol{X}$ to be completed, moving from right to left on each row. Rule 2 provides a check on the calculations. Further checks which have been discovered include the following:

5) $+1$ and -1 alternate along the last row of $\boldsymbol{X}$.

6) All the column sums of $\boldsymbol{X}$ are zero, except for the last column which adds to 2^{n-1}. The latter is a well-known property of binomial coefficients, and follows from rule 3. The final column elements of $\boldsymbol{X}$ are the coefficients in the binomial expansion of $(s+1)^{n-1}$, arranged in order of descending powers of s. Setting $s=1$ gives the stated result.

To illustrate these properties, the matrix $\boldsymbol{X}$ is shown below, for $n=5$, with the bent arrow indicating the sequence of calculation by rule 4.

$$\boldsymbol{X} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -4 & -2 & 0 & 2 & 4 \\ 6 & 0 & -2 & 0 & 6 \\ -4 & 2 & 0 & -2 & 4 \\ 1 & -1 & 1 & -1 & 1 \end{bmatrix}.$$

To complete the matrix $\boldsymbol{GF}$, each term x_{jn} is augmented by the amount

$$(-1)^j\frac{n!}{(n-j)!j!}\cdot\frac{p_n}{2} \qquad (j = 1, 2, \cdots, n).$$

The coefficients of $p_n/2$ are, in fact, the elements of the first column of $\boldsymbol{G}$. These are the coefficients in the binomial expansion of $(s-1)^n$, arranged in order of descending powers of s, but omitting the coefficient of s^n. By setting $s=1$ we see that the sum of such binomial coefficients is zero. The coefficients of $p_n/2$, therefore, add to -1.

In conclusion, attention is drawn to errors in the numerical example done in the original paper. The correct result is

$$\boldsymbol{b} = \text{column}\left(\frac{-13}{19}, \frac{42}{19}, \frac{-30}{19}, 1, \frac{-5}{19}\right).$$

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Manuscript received November 22, 1967.
[1] H. M. Power, *IEEE Trans. Education (Short Papers)*, vol. E-10, pp. 114–116, June 1967.

The Concept of "Zero Resistance"

***Abstract*—It is shown that conduction in an ideal, lossless conductor is different from conduction in a vanishingly small resistance and that it is not appropriate to use the term "zero resistance" to describe the lossless conductor.**

Levine has discussed[1] the paradox of a charged capacitor discharged through a zero-resistance conductor. Since a zero-resistance conductor is considered to be an ideal conductor, i.e., a lossless medium, what happens to the energy that is originally stored in the electric field of the capacitor? Levine points out that for the case of arbitrarily small, but nonzero, resistances the power dissipated in the resistance (as calculated from the voltage-current product) equals the rate at which the capacitor is losing energy so that energy is conserved. In the limiting case of zero resistance this power must still be dissipated in the resistor, a fact which is not consistent with the concept of the zero-resistance conductor being a lossless medium.

It is the purpose of this correspondence to point out that this paradox is created because of a falacious premise. Conduction in a lossless region is not the same as conduction in a region of vanishingly small resis-

Manuscript received April 1, 1968.
[1] R. C. Levine, "Apparent nonconservation of energy in the discharge of an ideal capacitor," *IEEE Trans. Education*, vol. E-10, pp. 197–202, December 1967.

tance. The power input to the electrons is indeed the product of potential difference and current but if the circuit element is "lossless," this power is not dissipated but is carried by the electrons in the form of kinetic energy. As an electron moves through the ideal conductor its kinetic energy will continually increase because of the existence of the field. This energy will not be dissipated in the conductor but will be given up in the form of heat when the electron strikes the positively charged capacitor plate.

A zero resistance could be made in principle in several ways: a) the interaction between the electrons and their surroundings could be vanishingly small, b) this interaction could be nonzero but the density of electrons could be infinitely large, or c) the cross-sectional area of the conductor could be infinitely large. Cases b) and c) would correspond to the limiting case discussed by Levine and would result in dissipation of energy in the conductor despite the zero resistance. Case a) would be lossless and would result in the increase of kinetic energy discussed in the preceding paragraph. The discharge of a capacitor through a zero resistance requires a finite displacement of mass (charge) in zero time. This is mathematically possible in the limiting cases b) and c) but is impossible in the case a) which is a correct representation of a lossless element. It is obvious from the brief description of conduction through a lossless medium given above that this mass transport will indeed take a finite time.

Ohm's law (from which the resistance concept derives) can be shown to be applicable only if the power transferred to the electrons by the field is transferred in turn from the electrons to the medium (e.g., crystal lattice) through which the electrons are moving. Conduction through the zero-resistance elements described as b) and c) could be considered to be following a limiting form of Ohm's law. Conduction through the lossless element a) would not be described by Ohm's law.

It can be seen that an ideal conductor constructed from a lossless medium does not have the properties of a vanishingly small resistance. For this reason it cannot be and should not be described by the term "zero resistance."

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Comments on "The Laplace Transformation of the Impulse Function For Engineering Problems"

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***Abstract*—Some confusion exists about the concept and properties of the impulse or delta function. It is shown that the proposed changes and qualifications of Etzweiler and Steele[1] are totally unnecessary. Instead, it is shown that if one considers carefully the series of events at 0^-, 0, 0^+, things are easily put in their proper perspective.**

Manuscript received December 15, 1967.

In the above short paper,[1] the authors have certainly brought out a subtle and often neglected point of linear analysis involving the transformation of the impulse function. However, no substantial justification exists in their statement of the Laplace transformation as

$$\mathcal{L}[f(t)] = F(s) = \lim_{\substack{\epsilon \to 0 \\ \epsilon > 0}} \int_{\epsilon}^{\infty} f(t)\epsilon^{-st}dt,$$

and furthermore, in the shifting of $\delta(t)$ from 0 to ξ. The above two modifications will, I think, confuse this delicate issue even further for the juniors, and they are totally unnecessary at any level of instruction, undergraduate or graduate.

I consider their example, as shown in Fig. 1. The apparent inconsistency in the evaluation of $I(s)$ by the two parallel approaches which Etzweiler and Steele present should not be sought in the statement of the Laplace transformation, but rather in the following fallacy.

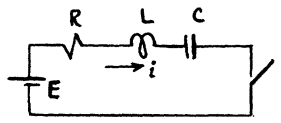

Fig. 1.

When the switch is closed at $t=0$, the current as a function of t, for $t \geq 0$, is given by

$$Eu(t) = Ri + L\frac{di}{dt} + \frac{1}{C}\int idt. \qquad (1)$$

As we differentiate the above equation,

$$\frac{dEu(t)}{dt} \neq E\delta(t) \quad \text{but} \quad \frac{dEu(t)}{dt} = 0 \quad \text{for } t \geq 0^+.$$

We cannot allow $u(t)$ to be anything other than 1, since the initial conditions introduced at $t=0^+$ take care of all the energy stored in the circuit at $t=0^+$. In other words, the effect of $\delta(t)$ at $t=0$ is accounted for by the initial conditions at $t=0^+$. The very presence of those initial conditions is due to the discontinuity of $u(t)$ at $t=0$. So,

$$I(s) \neq \frac{2E}{LS^2 + RS + \frac{1}{C}}$$

$$\text{but } I(s) = \frac{E}{LS^2 + RS + \frac{1}{C}}. \qquad (2)$$

Now, when we take the direct transform of (1), the initial condition

$$\frac{di}{dt}(0_+) = \frac{E}{L}$$

does not appear at all in the transformed equation, and we get

$$\frac{E}{S} = RI(s) + LSI(s) - Li(0_+) - \frac{1}{CS}I(s),$$

which leads to the solution (2), above.

Frequently, functions of time, such as the step function, encountered in linear system analysis have a discontinuity at $t=0$. In such cases the value of the function at the discontinuity is not uniquely defined and consequently differentiation at the discontinuity is not defined unless we step over on either side of the discontinuity. To "clear up the air" in the somewhat congested area of 0^-, 0, 0^+, ϵ (see Fig. 2). I have this suggestion: let's let the impulse function occur at 0 or anywhere else it happens to occur and formulate our integration from either 0^- or 0^+ to t. Either method will work well, as long as we adhere to it consistently.

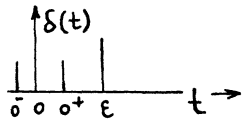

Fig. 2.

1) If we state the Laplace transform defining integral as

$$F(s) \triangleq \int_{0^-}^{\infty} f(t)e^{-st}dt,$$

the integration process will then include the full contribution of any impulse at the origin. The initial conditions will then be evaluated at $t=0^-$ and they will simply tell us about the history of the circuit (energy stored) before $\delta(t)$ happens at $t=0$.

2) If we state the Laplace transformation as

$$F(s) \triangleq \int_{0^+}^{\infty} f(t)e^{-st}dt,$$

the initial conditions at 0^+ will take care of the impulse function at 0. The second method perhaps is preferable for pedagogical purposes since most people are used to evaluating the initial conditions at $t=0^+$. Of course, if there is no impulse function at 0 then the initial conditions remain the same from 0^- to 0^+, and the two integral formulations above become identical.

Thus, a wider class of situations can be encompassed where the circuit already has been energized before $t=0^-$, and some change ($\delta(t)$ or other jump-off in the function) occurs in the excitation at $t=0$.

All of the above can be summarized, perhaps, by "move Mohammed to the mountain, not the mountain to Mohammed."

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***Abstract*—Defining the impulse as a narrow pulse of vanishingly small width provides a physical insight which is usually lacking in other definitions.**

[1] G. A. Etzweiler and S. A. Steele, *IEEE Trans. Education (Short Papers)*, vol. E-10, pp. 171–173, September 1967.

[2] Manuscript received January 10, 1968.