Review of Theories of Scattering of Elastic Waves by Cracks

EDGAR A. KRAUT

Abstract—The detection of cracks with the aid of ultrasonics is an important nondestructive evaluation (NDE) technique. The corresponding theoretical problem of the scattering of elastic waves by cracks has also been studied by scientists working in many different fields. Contributions to our knowledge of the subject have come from such diverse areas as geophysics, applied mathematics, electrical engineering, and continuum mechanics. Many of the results obtained by workers in those fields are also of interest to the NDE community and for that reason a review is presented here of current results and profitable directions for future research.

I. Introduction

THE ULTRASONIC detection of cracks in the interior of an elastic solid by the use of surface transducers is a fundamental Nondestructive Evaluation (NDE) problem. The presence of cracks may be detected either by observing the back scattered elastic waves using the launching transducer as a receiver or by observing obliquely scattered waves with a separate receiving transducer located elsewhere on the surface. Unfortunately, most of the theoretical work on the scattering of elastic waves from cracks has been confined to the case of a crack in an unbounded elastic solid, a situation far different from the experimental one. Even in that case, exact results are available only for a crack occupying a half plane. Exact results for cracks having a finite surface area, such as a penny shaped crack, are not available, although many approximate calculations have been published, particularly in the low-frequency limit. Before delving into the mathematical details of scattering from cracks, it is important to recognize that idealized cracks and real cracks may differ substantially in their behavior. For example, a cracked specimen may show different ultrasonic scattering characteristics depending on whether it is loaded or not. Such differences may be attributable to the closing of cracks under compression or the opening of cracks under tension. Important as such considerations are, they have received little attention from theorists. Consequently, until a better description of the boundary conditions at a crack becomes available, the idealized theory must be employed.

II. Idealized Description of Cracks

From the theoretical point of view, a crack is a two-dimensional surface of finite or infinite area located in the interior of an elastic solid. For example, a penny shaped crack can be thought of as the result of removing a thin disk shaped section of material from the interior of a solid. Boundary conditions are now applied at the surfaces of the void which has been created. In the case of a weak crack, the surfaces of the void are taken as free surfaces where the stress must vanish. In the case of a rigid crack, the void is imagined to be filled with a completely rigid material which pins the walls of the crack so that the displacement is zero everywhere. In each case, the finite thickness of the disk is neglected and both faces are thought of as occupying the same plane. This approach, while mathematically convenient, avoids the question of how the faces of the crack interact with one another and whether the crack is open or closed. Typical examples of two-dimensional cracks are shown in Figs. 1 and 2.

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The author is with the Science Center, Rockwell International, Thousand Oaks, Calif. 91360.

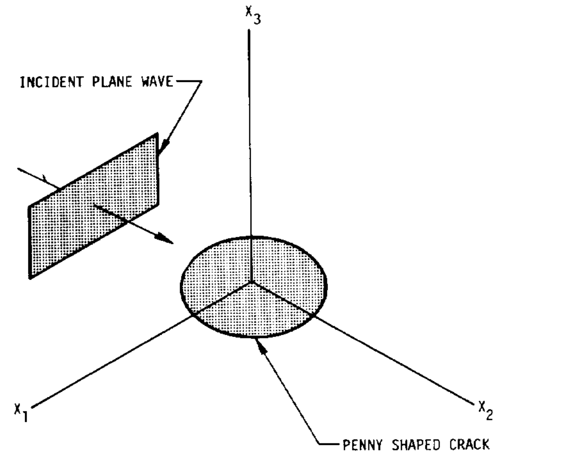

Fig. 1. Geometry of the scattering problem for a penny shaped crack.

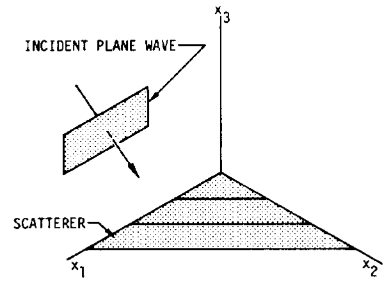

Fig. 2. Geometry of the scattering problem for a crack occupying a quarter plane.

III. Scattering of Elastic Waves by a Crack Occupying a Half Plane

There are two features of a crack that are essential in determining its behavior as a scatterer of elastic waves. The first is that a crack represents a two-dimensional surface across which

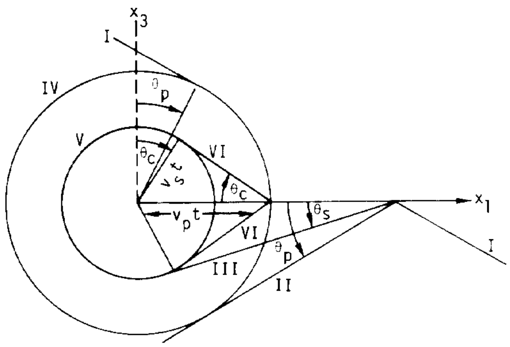

Fig. 3. Diffracted and reflected wavefronts produced by an incident compressional wave on a half-plane crack.

the stress or displacement (or both) can be discontinuous. The second is that cracks have edges which generate diffracted waves. The scattered waves produced by an incident compressional wave striking a half-plane crack are shown in Fig. 3. For the case of a weak crack, in addition to the wavefronts shown, there will also be a Rayleigh wave propagating away from the edge of the crack and confined to the plane of the crack. The first exact solution for scattering by a rigid half plane crack was given by Fridman [1] in 1948. The corresponding problem of the scattering of time harmonic shear and compressional waves by a weak half plane was solved exactly by Maue [2] in 1953. A very detailed exact treatment of the diffraction of incident shear and compressional pulses by weak and rigid half planes was presented by De Hoop [3] in 1958. The half-plane problem is particularly simple because displacements parallel to the edge of the half plane are decoupled, making the remaining problem two-dimensional. Since the exact solution is known, the half-plane problem provides a convenient means of testing various approximate solutions such as the Kirchoff approximation. Experimentally, scattering from a half plane can be modeled as shown in Fig. 4. This corresponds to the theoretical problem of scattering from a half plane at a finite depth beneath the free surface of an elastic half space. In addition to the types of scattered waves shown in Fig. 3 there is now also the possibility of multiple scattering, waveguiding, and the excitation of Lamb waves in the region between the half plane and the free surface. The corresponding theory does not seem to have been worked out in any detail, and because of the ease with which the problem can be modeled, further work seems worthwhile.

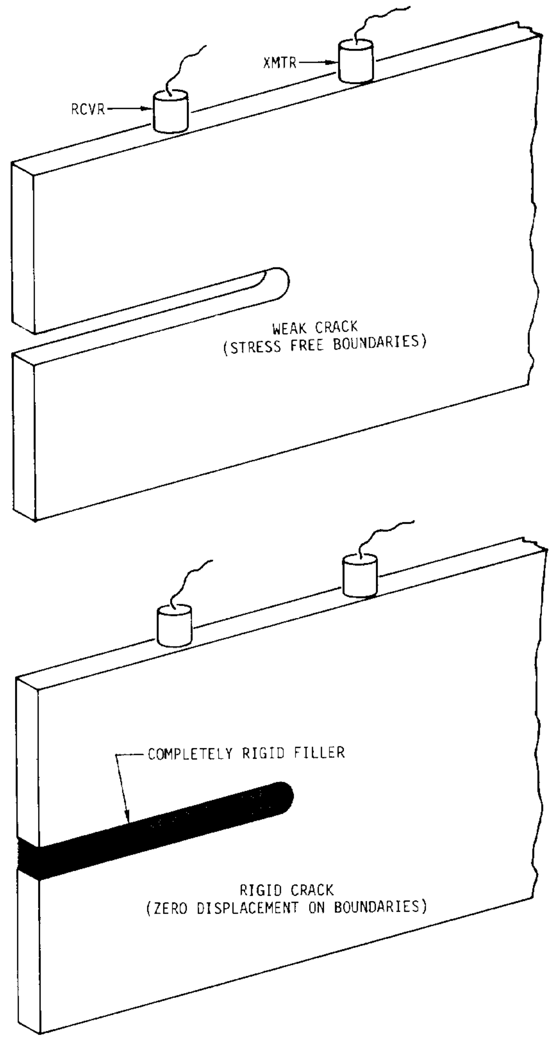

Fig. 4. Experimental models for scattering and diffraction from stress free and rigid half-plane cracks.

IV. Scattering of Elastic Waves by a Two Dimensional Crack of Arbitrary Shape in an Unbounded Elastic Medium

Consider the scattering of an arbitrary incident wave by a crack Σ of finite extent and vanishing thickness. In an elastic solid Σ is a two-dimensional region across which the displacement and stress may be discontinuous. Both the incident and scattered elastic waves satisfy the homogeneous wave equation

$$c_{ijpq}\left(\frac{\partial^2 u_p}{\partial x_j \partial x_q}\right) - \rho \frac{\partial^2 u_i}{\partial t^2} = 0. \tag{4.1}$$

With the aid of Green's theorem, the scattered displacement field u_i^s can be expressed in terms of the jumps across Σ in the displacement and stress. Let n_i^+ and n_i^- denote the unit vectors in the direction of the normal to Σ^+ and Σ^-, respectively; Σ^+ is one face of Σ and Σ^- is the other face. The positive sense of n_i^+ and n_i^- is taken towards Σ; hence, $n_i^+ = -n_i^-$. The jump in displacement is given by

$$[u_i]_-^+ = u_i^+ - u_i^- \tag{4.2}$$

and the jump in the stress is given by

$$[\tau_{ij}]_-^+ n_j^+ = (\tau_{ij}^+ - \tau_{ij}^-)\, n_j^+ \tag{4.3}$$

where

$$\tau_{ij} = c_{ijpq} \frac{\partial u_p}{\partial \xi_q}.$$

In the time harmonic case the scattered field $u_i^s(x_1, x_2, x_3, \omega) e^{i\omega t}$ is given by [3]

$$u_i^s(x_1, x_2, x_3, \omega) = \int_\Sigma c_{jkpq} G_{ij} \left[\frac{\partial u_p}{\partial \xi_q}\right]_-^+ n_k^+ \, ds + \left(\frac{\partial}{\partial x_q}\right) \int_\Sigma c_{jkpq} G_{ip} [u_j]_-^+ n_k^+ \, ds. \quad (4.4)$$

The first term on the right-hand side of (4.4) is the displacement due to a single layer distribution on Σ. The second term is the displacement due to a double layer distribution on Σ. The term due to the single layer distribution leads to a displacement which is continuous across Σ but gives a stress which jumps across Σ by the assumed amount. On the other hand, the term due to the double layer distribution leads to a displacement which jumps across Σ by the assumed amount but gives a continuous stress across Σ. The tensor Green's function G_{ij} for the elastic wave equation (4.1) is well known [3] and is given by

$$G_{ij}(x - \xi, \omega) = \frac{1}{4\pi\rho} \left\{ \frac{1}{\omega^2} \frac{\partial^2}{\partial x_i \partial x_j} \cdot \left\{ \frac{\exp(-ik_s r)}{r} - \frac{\exp(-ik_p r)}{r} \right\} + \frac{1}{v_s^2} \frac{\exp(-ik_s r)}{r} \delta_{ij} \right\} \quad (4.5)$$

where

$$r = \{(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2 + (x_3 - \xi_3)^2\}^{1/2},$$

$$k_p = \omega/v_p, \quad k_s = \omega/v_s, \quad \rho v_p^2 = \lambda + 2\mu, \quad \text{and} \quad \rho v_s^2 = \mu.$$

The unknown stress and displacement discontinuities in the integrands of (4.4) are determined by applying boundary conditions to the scattered field $u_i^s(x_1, x_2, x_3, \omega)$ at the surface of the crack. The resulting integral equations have been solved exactly for the case where Σ is a half plane [1]-[3]. For cracks of more general shape, only approximate solutions are known.

V. Kirchoff's Approximation

In the Kirchoff approximation, specific assumptions are made about the jumps in the stress and displacement across a crack. This permits the direct evaluation of (4.4) without having to solve any integral equations. We are interested here in the errors introduced by this approximation. Kirchoff's original theory assumes that the wave function and its normal derivative are determined entirely by the incident wave on the geometrically illuminated part of the scatterer. The wave function and its normal derivative are assumed to vanish on the dark part of the scatterer. The corresponding assumptions for the scattering of elastic waves are that the amounts by which the stress and displacement jump across a crack are numerically equal to the corresponding values of the incident wave at the illuminated surface of the crack. When this assumption is made it is found [3] that all the reflected waves are lost in the solution. The critical angle head wave is also lost. Only the incident wave and the diffracted waves generated by the edge of the crack are obtained. This explains why the Kirchoff assumptions are supposed to solve diffraction by a perfectly absorbing scatterer (in optical terms a black screen). The Kirchoff approximation can be modified in such a way that the correct reflected waves are obtained [3], [4], however, the critical angle head waves are always lost. One can think of the Kirchoff approximation as a method of specifying the physical properties of a crack in terms of the jumps in displacement and stress across it. If these jumps are numerically equal to the corresponding values of the incident wave at the crack surface, the crack is perfectly absorbing or "black." In general, however, the theories of Kirchoff and modifications of it are poor substitutes for rigorous diffraction theory (wave equation plus boundary conditions) because they do not correctly describe the field in the vicinity of the scatterer and in the long-wavelength limit because they entirely fail to predict the correct order of magnitude of the field far from the scatterer [4].

VI. Scattering from a Penny Shaped Crack

The scattering of elastic waves by a penny shaped crack in an unbounded elastic solid has been treated by several investigators [5]-[15]. Some of the earliest work is that of Filipczynski [5] (1961) who treated the problems by separation of variables in an axially symmetric oblate spheroidal coordinate system. Use of that coordinate system permits a simple statement of the boundary conditions on the surface of a disk since a disk is one of the coordinate surfaces. The case considered by Filipczynski [5] is that of the scattering of a normally incident plane compressional wave by a disk in the limit in which the radius of the disk is much shorter than the wavelength of the incident wave (long wavelength or Rayleigh limit $ka \to 0$). In this case, the Kirchoff approximation is expected to be poor. Far away from the disk in Fig. 1, the reflected waves can be referred to a set of spherical coordinates

$$x_3 = R\cos\theta \qquad x_2 = R\sin\theta\sin\phi \qquad x_1 = R\sin\theta\cos\phi. \quad (6.1)$$

When a plane compressional wave is normally incident on the penny shaped crack in Fig. 1, then axial symmetry permits the scattered displacement field to be represented by the gradient of a scalar potential φ and the curl of a vector potential having only one component $\vec{A} = \vec{e}_\phi A_\phi$ so that,

$$\vec{U} = \nabla\varphi + \nabla x(\vec{e}_\phi A_\phi). \quad (6.2)$$

Filipczynski's expression [5] for φ and A_ϕ in the long-wavelength far-field limit are

$$\varphi = \varphi_0 \frac{e^{-ik_L R}}{R} \cos\theta \quad (6.3)$$

$$A_\phi = A_0 \frac{e^{-ik_T R}}{R} \sin\theta \tag{6.4}$$

where

$$\varphi_0 = -\frac{2a}{3\pi} \frac{(k_L a)^2}{1 + \frac{3}{4}\left(\frac{k_T}{k_L}\right)^2} \tag{6.5}$$

$$A_0 = -\frac{2a}{3\pi} \frac{(k_T a)^2}{1 + \frac{3}{4}\left(\frac{k_T}{k_L}\right)^2}. \tag{6.6}$$

The longitudinal and transverse wavenumbers are k_L and k_T, respectively. The corresponding displacement components u_R, u_θ, and u_ϕ to order R^{-1} are

$$u_R = -ik_L\varphi_0 \frac{e^{-ik_L R}}{R} \cos\theta \tag{6.7}$$

$$u_\theta = ik_T A_0 \frac{e^{-ik_T R}}{R} \sin\theta \tag{6.8}$$

$$u_\phi = 0. \tag{6.9}$$

Different expressions for the far-field displacements due to diffraction of elastic waves by rigid and weak circular disks have been obtained by Mal [7]-[11], and formulas for the corresponding scattering cross sections appear in the works of Robertson [6] and Filipczynski [5]. Details of the computation of the scattering cross section when plane time harmonic compressional or shear waves are incident on two- or three-dimensional obstacles in an infinite elastic solid have been discussed by several authors [16]-[19]. Except for Filipczynski's work, all the other cited results have been obtailed in cylindrical coordinates by iteratively solving integral equations for the scattered field in the long-wavelength limit. The work of Roberston [6] is particularly interesting. He assumed that the scattered field due to a plane compressional wave normally incident on a penny shaped crack could be modeled by a harmonically oscillating piston on the surface of a semi-infinite elastic solid. He thus replaced the problem of calculating the scattered field from a disk shaped flaw in an unbounded elastic medium by the problem of calculating the radiation field of a disk shaped transducer on the surface of an elastic half space. Robertson considered the case where a time harmonic normal stress is prescribed at the disk surface and the displacements are zero elsewhere on the boundary [6]. He also treated the complimentary case where the displacement is prescribed at the disk surface and the stresses are zero elsewhere on the boundary [20]. In both of these cases, the fact that stress is prescribed over one portion of the boundary and the displacement is prescribed over the remaining portion leads to integral equations which have only been solved approximately in the long-wavelength limit.

There is another kind of disk shaped transducer problem that has been solved exactly by Miller and Pursey [21]. It is the problem of the field due to an oscillating normal stress applied over a disk shaped region on an otherwise free surface of a semi-infinite elastic solid. Since in this case the stress alone is specified on the boundary, the problem can be solved exactly. The solution obtained by Miller and Pursey [21] given in terms of the potentials $\phi(r, z)$ and $\psi(r, z)$ is

$$\phi(r,z) = \frac{a}{\mu}\int_0^\infty \frac{(2k^2 - k_\beta^2)}{F(k)} \exp(-\nu z)\, J_1(ka)\, J_0(kr)\, dk$$

$$\psi(r,z) = \frac{a}{\mu}\int_0^\infty \frac{2\nu}{F(k)} \exp(-\nu' z)\, J_1(ka)\, J_0(kr)\, dk \tag{6.10}$$

where

$$\nu = \sqrt{k^2 - k_\alpha^2}$$

$$\nu' = \sqrt{k^2 - k_\beta^2}$$

$$F(k) = (2k^2 - k_\beta^2)^2 - 4k^2\nu\nu' \tag{6.11}$$

and k_α, k_β are the compressional and shear wavenumbers, respectively. The radius of the disk is $r = a$ and the modulus of rigidity of the solid is μ. The displacements u_r, *and* u_z and the stresses T_{rz} and T_{zz} are related to the potentials by

$$u_r = \frac{\partial\phi}{\partial r} + \frac{\partial^2\psi}{\partial r\partial z}$$

$$u_z = \frac{\partial\phi}{\partial z} + \left(\frac{\partial^2}{\partial z^2} + k_\beta^2\right)\psi$$

$$T_{rz} = \mu\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right)$$

$$T_{zz} = -\lambda k_p^2\phi + 2\mu\frac{\partial u_z}{\partial z}. \tag{6.12}$$

The boundary conditions satisfied by this solution are

$$[T_{zz}]_{z=0} = 1, \quad \text{for} \quad r \leqslant a$$

$$[T_{zz}]_{z=0} = 0, \quad \text{for} \quad r > a$$

$$[T_{rz}]_{z=0} = 0, \quad \text{for} \quad 0 \leqslant r < \infty. \tag{6.13}$$

In the spherical coordinate system of (6.1), the far field asymptotic form of the solution valid for large R and small a is given by

$$u_R = -\frac{a^2}{2\mu}\frac{e^{-iR}}{R}\frac{\cos\theta(\xi^2 - 2\sin^2\theta)}{F_0(\sin\theta)} \tag{6.14}$$

$$u_\theta = -\frac{ia^2\xi^3}{2\mu}\frac{e^{-i\xi R}}{R}\frac{\sin 2\theta(\xi^2\sin^2\theta - 1)^{1/2}}{F_0(\xi\sin\theta)} \tag{6.15}$$

where the compressional wavenumber k_α has been replaced by unity in (6.14) and the new shear wavenumber ξ is given by $\xi = k_\beta/k_\alpha$. The function $F_0(\zeta)$ is defined by

$$F_0(\zeta) = (2\zeta^2 - \xi^2)^2 - 4\zeta^2(\zeta^2 - 1)^{1/2}(\zeta^2 - \xi^2)^{1/2}. \tag{6.16}$$

The corresponding expressions obtained by Mal [7], [11] for the asymptotic scattered fields due to rigid and weak disks in unbounded elastic solids are

$$u_R = -\frac{i\cos\theta}{\rho\alpha^2} P(k_\alpha\sin\theta)\frac{e^{-ik_\alpha R}}{R}$$

$$u_\theta = \frac{i\sin\theta}{\rho\beta^2} P(k_\beta\sin\theta)\frac{e^{-ik_\beta R}}{R} \tag{6.17}$$

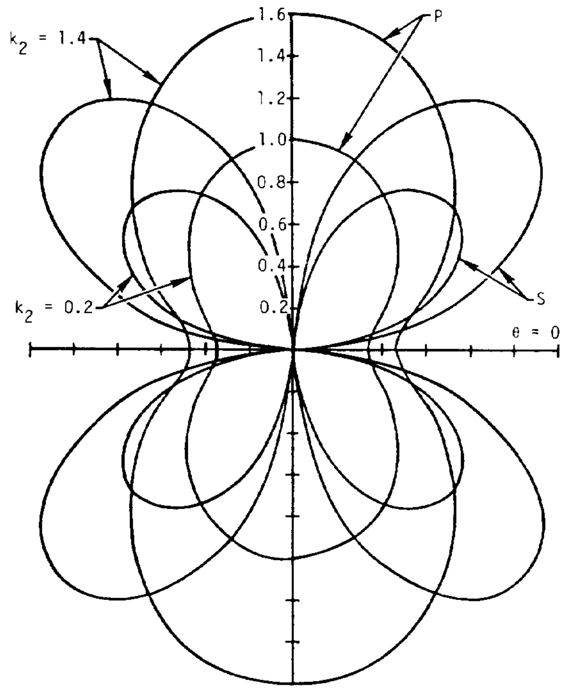

Fig. 5. Radiation field of the diffracted compressional and shear waves produced by a normal incident compressional wave on a penny shaped crack. (From Mal [11].)

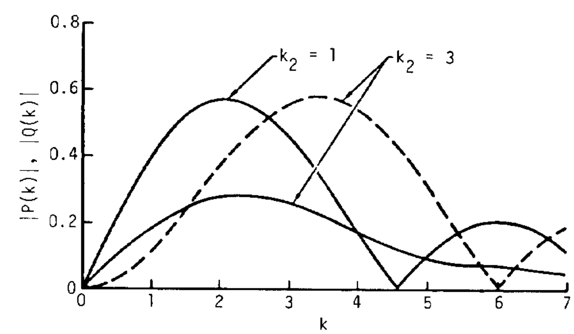

Fig. 6. Amplitude factors for the far field radiation pattern produced by scattering from a penny shaped crack. (From Mal [11].)

for rigid disks [7], and

$$u_R = \text{const.}\,\frac{e^{ik_\alpha R}}{R}\,(\xi^2 - 2\sin^2\theta)\,\frac{P(k_\alpha \sin\theta)}{\sin\theta}$$

$$u_\theta = \text{const.}\,\frac{e^{ik_\beta R}}{R}\,\sin 2\theta\,\frac{P(k_\beta \sin\theta)}{\sin\theta} \tag{6.18}$$

for weak disks [11].

The amplitude factor $P(k)$ is obtained from the approximate or numerical solution of the integral equations for the scattered field. In these expressions, ρ is the density of the solid and α and β are the compressional and shear wave velocities, respectively. The far field radiation pattern and the amplitude factor $P(k)$ obtained by Mal [11] are shown in Figs. 5 and 6.

VII. Related Diffractional and Scattering Problems

The diffraction of plane elastic waves by two-dimensional straight strips or cracks of finite width has been treated by Ang and Knopoff [22], [23] and by Loeber and Sih [24], [25]. More recently, Keer and Luong [26] have considered the diffraction of waves and stress intensity factors in cracked layered composites. Nonaxisymmetric scattering of plane compressional elastic waves by a rigid disk has also been examined by Datta [27]. Related numerical work using the finite-element approach to acoustic scattering from elastic and rigid disks immersed in water has recently been published by Hunt *et al.* [28]. Their results complement those of Ermolov [29] and Cohen [30] based on analytical approximations to acoustic scattering by disks. A comprehensive review of acoustic (and electromagnetic) scattering from disks and other simple shapes has been compiled by Bowman, Senior, and Uslenghi [31] and some useful information on the diffraction of elastic waves is contained in a selected review by Pao and Mow [32]. In addition to the problems of scattering of elastic waves from half-planes, disks, and strips, another two-dimensional scattering surface of interest is a crack occupying a quarter plane, i.e., one quadrant of an (x, y) plane. This problem and the solution of related integral equations has been discussed in a series of papers by Kraut [33]-[36]. Unfortunately, an exact analytical solution of the quarter-plane scattering problem for elastic waves does not appear to be possible.

VIII. Conclusion

In a brief review such as this a great many topics of current interest in elastic wave propagation have to be omitted. These include recent advances in finite difference [37], [38] and finite element methods [39], application of Keller's geometric theory of diffraction [40], [41] to elastic wave propagation problems [42], first motion methods in the scattering of elastic pulses [43], variational methods [44], numerical solution of integral equations arising in scattering problems, applications of the J.W.K.B. and Born approximations to elasticity, use of the Watson transformation [31], asymptotic expansions, perturbation methods, as well as long, short, and intermediate wavelength approximations in general.

Many problems of interest to NDE involve scattering from cracks and flat bottom holes in bounded or semi-infinite elastic solids as opposed to unbounded solids. The presence, in addition to a crack, of one or more extra free surfaces greatly complicates the mathematics of the scattering problem. The development of effective approximate methods to solve such problems and comparison of the results obtained with experiment can contribute significantly to progress in NDE.

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